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Diffusing Passive Tracers in Random Incompressible Flows: Statistical Topography Aspects

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The paper studies statistical characteristics of the passive tracer concentrations and of its spatial gradient, in random incompressible velocity fields from the viewpoint of statistical topography. The statistics of interest include mean values, probability distributions, as well as various functionals characterizing topographic features of tracers. The functional approach is used. We consider the influence of the mean flow (the linear shear flow) and the molecular diffusion coefficient on the statistics of the tracer. Most of our analysis is carried out in the framework of the delta-correlated (in time) approximation and conditions for its applicability are established. But we also consider the diffusion approximation scheme for finite correlation radius. The latter is applied to a diffusing passive tracer that undergoes sedimentation in a random velocity field.

KEY WORDS: Diffusion; mean field; correlation function; Furutsu–Novikov formula; Markov process; long-normal probability law; correlation splitting; delta-correlated approximation; diffusion approximation; random topography.

1. INTRODUCTION

The study of passive tracer (or passive scalar) transport in random velocity flows is a classical topic in statistical fluid mechanics. Its applications range from questions of environmental pollutant diffusion in a turbulent atmosphere to problems of advection of heat and salinity in oceanic

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currents.[1-5] and from diffusion in porous media[6] to questions of the large-scale mass distribution in the late stages of the universe.[7,8]

The problem has been studied since the end of the 1950s, beginning with the pioneering work of Batchelor *et al.*[9,10] Many researchers (see, e.g., ref. 11–15) have obtained equations describing the statistical characteristics of the passive tracer field, both in Eulerian and in Lagrangian descriptions. This research activity continues vigorously at present.

An initially smooth tracer concentration $q(\mathbf{r}, t)$ which undergoes diffusion in a random velocity field acquires in time a complex spatial structure. For example, individual realizations of 2D fields often resemble a complex mountain landscape with randomly distributed peaks, valleys, saddles, and ridges, all of which evolve in time. The mean values such as statistical moments $\langle q(\mathbf{r}, t)\rangle$ and $\langle q(\mathbf{r}_1, t)\, q(\mathbf{r}_2, t)\rangle$, where $\langle .,.\rangle$ denotes averaging over the ensemble of realizations of the random velocity field, smooth out fine details. Such averaging usually brings forth spatiotemporal scales of the whole tracer domain while neglecting its fine dynamics. The detailed structure of the tracer field can be described, as in standard topographic maps,[16,17] in terms of level curves of the concentration field (2D case) or level surfaces (3D case)

$$q(\mathbf{r}, t) = q \text{ const}$$

Alternatively, and often more conveniently, we shall employ the distributional (indicator) function

$$\Phi_{t,\mathbf{r}}(q) = \delta(q(\mathbf{r}, t) - q)$$

which has "values" concentrated on the level curve (surface). Figure 1 shows schematically the numerical simulation of the time evolution of the level curve $q(\mathbf{r}, t) = \text{const}$ of the 2D concentration field.[4] In view of the incompressibility of the fluid flow, the area bounded by the level curve is conserved, but the picture clearly becomes increasingly fragmented; we observe both the steepening of gradients and the contour dynamics at progressively smaller scales. With help of the distributional indicator function one can study the dynamics of various functionals of level curves (surfaces).

For example, integrating the norm of tracer gradient over the level set

$$A_0(t) = \int d\mathbf{r}\, |\mathbf{p}(\mathbf{r}, t)|\, \delta(q(\mathbf{r}, t) - q) = \oint dl$$

[4] It was obtained by Dr. Yiming Hu while he was to replicating computer experiments of ref. 18.

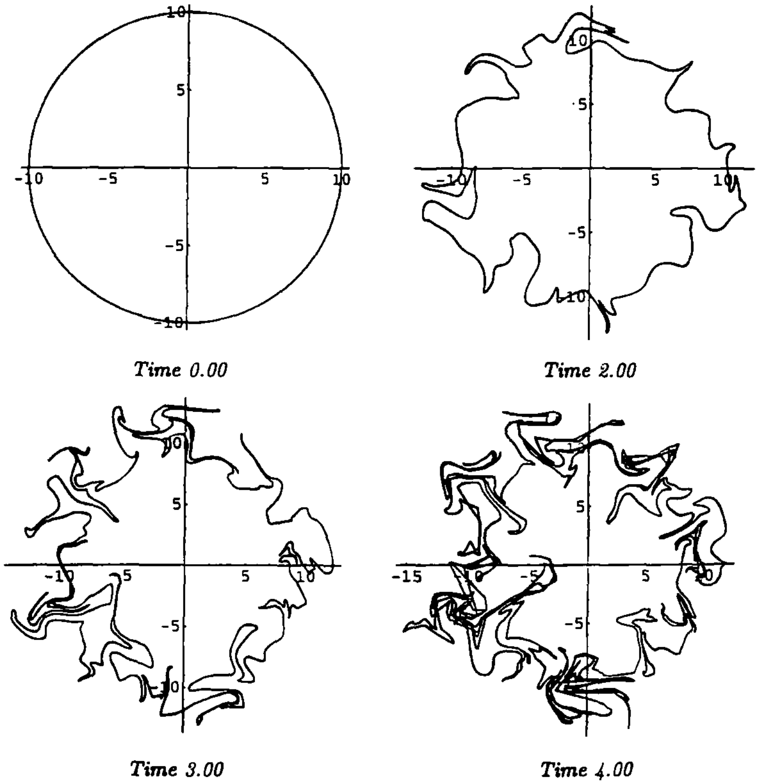

Fig. 1. Numerical simulation of time evolution of the level curve $q(\mathbf{r}, t) = \text{const}$ of the 2D concentration field in an incompressible flow.

we get its arc-length in the 2D case and the level surface area in the 3D case.[19,21] On the other hand, the integral

$$S(t, q) = \pm \tfrac{1}{2} \int d\mathbf{r}\, \mathbf{r}\mathbf{p}(\mathbf{r}, t)\, \delta(q(\mathbf{r}, t) - q)$$

gives the area enclosed by the level contour and

$$\mathcal{N}(q, t) \leqslant \int_0^\infty dr\, \frac{|\mathbf{r}\mathbf{p}(\mathbf{r}, t)|}{r}\, \delta(q(\mathbf{r}, t) - q), \qquad r = |\mathbf{r}|$$

estimates the number of connected level components as they evolve in time.

Note, that averaging indicator functions

$$P_{t,\mathbf{r}}(q) = \langle \delta(q(\mathbf{r}, t) - q)$$

$$P_{t,\mathbf{r}}(q, \mathbf{p}) = \langle \delta(q(\mathbf{r}, t) - q)\, \delta(\mathbf{p}(\mathbf{r}, t) - \mathbf{p}) \rangle$$

over velocity ensemble defines, respectively, the one-point probability density of the random field q and the joint probability density of the tracer and its spatial gradient. In this fashion, even one-point statistical characteristics of the tracer permit us to determine statistical means of various functionals of the above types and make statements about the dynamics of individual tracer realizations in a random velocity field. This is particularly useful for problems of passive tracer diffusion in the atmosphere and the ocean, where, typically, one does not deal with ensembles, but rather individual realizations. The study of such problems constitutes the subject matter of *statistical topography*.

This picture is well exemplified by the tracer dynamics and can be demonstrated by very simple statistical models of the velocity field. For example, it is relatively easy to write down equations for the statistics of a passive tracer in the so-called *delta-correlated* random velocity field (see, e.g., ref. 22), which can be seen as an approximation to other, more realistic situations. There, a Lagrangian particle behaves like an ordinary Brownian particle.

The term *statistical topography* is widely used in the physical literature,[16, 20] but in the mathematics community related problems have been extensively studied within the theory of *random surfaces* or the *geometry of random fields*.[17] However, the latter has emphasized static geometric properties of classical "probabilistically" defined random fields like Brownian sheets, spatially homogeneous fields, etc., whereas the main interest in the physics community has been on "dynamically" defined random fields, that is on random fields satisfying certain partial diferential equations.

In this paper, we use a *functional approach* to study the problem of passive tracer diffusion in random velocity fields from the viewpoint of statistical topography. Both the general set up, and approximate methods permitting efficient numerical computations are considered. Applicability conditions for the latter are formulated.

The papers is constructed as follows. Section 2 sets up the general problem and, in the absence of molecular diffusion, provides equations for the tracer distribution functions in Lagrangian and Eulerian descriptions. We also establish the relation between the two descriptions. These results, however, are valid only for a finite time interval.

Section 3 formulates the general functional approach which permits an efective analysis of Gaussian velocity field fluctuations, the so-called Furutsu–Novikov formalism).

The subsequent analysis of the problem in the case of the delta-correlated (in time) velocity field is provided in Section 4. Here we study the mean tracer concentration field and its correlation function which characterize the global space-time scales from the viewpoint of statistical topography. We also investigate the role of molecular diffusion and its characteristics. If the molecular diffusion is absent we obtain Fokker–Planck equations for the joint one-point probability density of the tracer concentration field and its spatial gradient. In particular, we show that the gradient norm has a log-normal distribution and conclude that its moments grow exponentially in time. Based on the Fokker–Planck equation we also calculate the time evolution of statistical characteristics of certain functions on level contours in the 2D case and level surfaces in the 3D case. In particular, the mean contour length is determined and an upper estimate for the mean number of connected components of the contour is found. In some cases they grow exponentially in time.

Furthermore, using the Fokker–Planck equation we study the influence of the drift on the tracer statistical characteristics. We show that the large drift gradient of the mean flow strengthens the role of weak velocity fluctuations in an exponential manner. The resulting Fokker–Planck equation is valid, however, only over the finite time interval. The interval size is estimated and is shown to depend on the molecular diffusion coefficient logarithmically.

The delta-correlated case precludes many special features connected with the finite correlation radius. The latter could be approached via the diffusion approximation method. We find conditions for its applicability and describe a number of subtle new effects. In particular, in Section 5 we develop the diffusion approximation scheme for a sedimentation problem in a random velocity field. We show that taking into account the finite range of the temporal velocity correlations leads to anisotropy of the effective diffusion coefficients with respect to the sedimentation direction.

All the approximations considered in this paper can be viewed as short-time correlation approximations. Physically, the assumption is that the velocity fluctuations have little effect on tracer statistics on time scales comparable to the temporal correlation radius.

The paper presents a novel approach to the classical problem of diffusing passive tracer in random velocity fields based on ideas and concepts of statistical topography. Here we apply them to stochastic tracer dynamics and get some new spatiotemporal characteristics of its evolution.

2. EVOLUTION OF PASSIVE TRACER CONCENTRATION

In this Section we formulate the dynamical problem in the Lagrangian and Eulerian descriptions, establish their connection, and prepare the ground for the statistical analysis of mean concentration and its correlation on the one hand, and the probability distribution of the tracer concentration and its spatial gradient on the other.

The basic equation that describes the evolution of the passive tracer density $q(\mathbf{r}, t)$ has the form

$$\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r}, t)\frac{\partial}{\partial \mathbf{r}}\right) q(\mathbf{r}, t) = \kappa \frac{\partial^2}{\partial \mathbf{r}^2} q(\mathbf{r}, t), \qquad q(\mathbf{r}, 0) = q_0(\mathbf{r}) \tag{1}$$

where κ denotes the "molecular" diffusion coefficient. Depending on the physical context, q could represent such quantities as temperature and salinity, of interest in oceanography, or, in the case of incompressible flows ($\operatorname{div} \mathbf{V} = 0$) below, it could also cover "matter" concentrations, such as air pollutants and oil droplets in an oil slick.

For incompressible flows, Eq. (1) has the form of a conservation law, the quantity

$$Q = \int d\mathbf{r}\, q(\mathbf{r}, t) = \int d\mathbf{r}\, q_0(\mathbf{r})$$

being conserved.

We assume velocity field $\mathbf{V}$ to be random with finite expectations and decompose it into the mean component

$$\mathbf{v}(\mathbf{r}, t) = \langle \mathbf{V}(\mathbf{r}, t) \rangle$$

and the random fluctuation

$$\mathbf{F}(\mathbf{r}, t) = \mathbf{V}(\mathbf{r}, t) - \mathbf{v}(\mathbf{r}, t)$$

Although Eq. (1) is linear, the equations for powers $q^n(\mathbf{r}, t)$ of interest to us are nonlinear:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r}, t)\cdot\frac{\partial}{\partial \mathbf{r}}\right) q^n(\mathbf{r}, t)\\ &\quad = \kappa \frac{\partial^2}{\partial \mathbf{r}^2} q^n(\mathbf{r} \cdot t) + \kappa n(n-1)\, q^{n-2}(\mathbf{r}, t)\, \mathbf{p}^2(\mathbf{r}, t) + n q^{n-1}(\mathbf{r}, t)\, \Delta q \end{aligned} \tag{1'}$$

They involve the spatial gradient $\mathbf{p}(\mathbf{r}, t) = \partial q(\mathbf{r}, t)/\partial \mathbf{r}$ of the tracer field.

Equation (1) gives the *Eulerian description* of the system.

A direct study of the probability distribution of $q(\mathbf{r}, t)$ is not possible if (1) contains the second-order (diffusion) term in $\mathbf{r}$. One point of interest is the limiting behavior of solutions as $\kappa \to 0$. Here, we require the initial tracer concentration $q_0(\mathbf{r})$ and its gradient $\mathbf{p}_0(\mathbf{r})$ to be large scale (the precise meaning will be explained later). Then, one can drop terms containing κ in (1), and consider the transport problem described by

$$\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r}, t) \cdot \frac{\partial}{\partial \mathbf{r}}\right) q(\mathbf{r}, t) = 0, \qquad q(\mathbf{r}, 0) = q_0(\mathbf{r}) \tag{2}$$

The dynamic equation (2) is physically relevant only over a limited time interval.

For a more complete statistical analysis in this time interval it is necessary to include the gradient field $\mathbf{p}(r, t) = \partial q(\mathbf{r}, t)/\partial \mathbf{r}$, which obeys

$$\begin{gathered}\left(\frac{\partial}{\partial t} + \mathbf{v}(\mathbf{r}, t) \frac{\partial}{\partial \mathbf{r}}\right) p_i(\mathbf{r}, t) = -\frac{\partial V_k}{\partial r_i} p_k(\mathbf{r}, t) \\ \mathbf{p}(\mathbf{r}, 0) = \mathbf{p}_0(\mathbf{r}) = \frac{\partial}{\partial \mathbf{r}} q_0(\mathbf{r})\end{gathered} \tag{3}$$

as a consequence of (2). Here the repeated indices designate, as usual, summation over them.

Let us introduce a distributional (indicator) function

$$\Phi_{t, \mathbf{r}}(q, \mathbf{p}) = \delta(q(\mathbf{r}, t) - q)\, \delta(\mathbf{p}(\mathbf{r}, t) - \mathbf{p}) \tag{4}$$

which determines the joint one-point probability distribution of fields q and p at a given spatial point in the Eulerian coordinates. We shall also consider more general two-point distribution of the tracer concentration field and its gradient

$$\begin{aligned}&\Phi_{t, \mathbf{r}_1, \mathbf{r}_2}(q_1, \mathbf{p}_1; q_2, \mathbf{p}_2) \\ &\quad = \delta(q(\mathbf{r}_1, t) - q_1)\, \delta(\mathbf{p}(\mathbf{r}_1, t) - \mathbf{p}_1)\, \delta(q(\mathbf{r}_2, t) - q_2)\, \delta(\mathbf{p}(\mathbf{r}_2, t) - \mathbf{p}_2) \\ &\quad \equiv \Phi_{t, \mathbf{r}_1}(q_1, \mathbf{p}_1)\, \Phi_{t, \mathbf{r}_2}(q_2, \mathbf{p}_2)\end{aligned} \tag{4'}$$

The additional information contained in (4'), would allow us to analyze various functionals of fields q and $\mathbf{p}$.

Based on Eqs. (2) and (3), one can easily obtain the dynamic evolution of functions (4) and (4′). In particular, $\Phi_{t,\mathbf{r}}(q, \mathbf{p})$ satisfies the Liouville equation[22]

$$\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r}, t) \cdot \frac{\partial}{\partial \mathbf{r}}\right) \Phi_{t,\mathbf{r}}(q, \mathbf{p}) = \frac{\partial V_k(r, t)}{\partial r_i} \frac{\partial}{\partial p_i} (p_k \Phi_{t,\mathbf{r}}(q, \mathbf{p})) \tag{5}$$

with the initial condition

$$\Phi_{0,\mathbf{r}}(q, \mathbf{p}) = \delta(q_0(\mathbf{r}) - q)\, \delta(\mathbf{p}_0(\mathbf{r}) - \mathbf{p})$$

The first-order partial differential equations (2) and (3) can be solved by the method of characteristics

$$\frac{d\mathbf{r}(t \mid \xi)}{dt} = \mathbf{V}(\mathbf{r}, t), \qquad \mathbf{r}(0 \mid \xi) = \xi \tag{6}$$

Then, (2) and (3) are reduced to the initial value problem for ODE's

$$\begin{aligned} &\frac{d}{dt} q(t \mid \xi) = 0, \qquad q(0 \mid \xi) = q_0(\xi) \\ &\frac{d}{dt} p_i(t \mid \xi) = -\frac{\partial V_k(\mathbf{r}, t)}{\partial r_i} p_k(t \mid \xi), \qquad p_i(0 \mid \xi) = \frac{q_0(\xi)}{\partial \xi_i} \end{aligned} \tag{6′}$$

along the characteristic curves. Equations (6) and (6′) give a closed form *Lagrangian description* of the system. Here and elsewhere $(\ldots \mid \xi)$ indicates conditioning by the initial marker ξ in the Lagrangian formulation. Solution q remains constant along characteristics, so

$$q(t \mid \xi) \equiv q_0(\xi)$$

Introducing a distributional (indicator) function

$$\Phi_t(\mathbf{r}, q, \mathbf{p} \mid \xi) = \delta(\mathbf{r}(t \mid \xi) - \mathbf{r})\, \delta(q(t \mid \xi) - q)\, \delta(\mathbf{p}(t \mid \xi) - \mathbf{p}) \tag{7}$$

which determines the joint density of particle distribution, we can write a similar Liouville equation

$$\left(\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{r}, t) \cdot \frac{\partial}{\partial \mathbf{r}}\right) \Phi_t(\mathbf{r}, q, \mathbf{p} \mid \xi) = \frac{\partial V_k(\mathbf{r}, t)}{\partial r_i} \frac{\partial}{\partial p_i} (p_k \Phi_t(\mathbf{r}, q, \mathbf{p} \mid \xi)) \tag{7′}$$

with the initial condition

$$\Phi_0(\mathbf{r}, q, \mathbf{p} \mid \xi) = \delta(\xi - \mathbf{r})\, \delta(q_0(\xi) - q)\, \delta\left(\frac{\partial q_0(\xi)}{\partial \xi} - \mathbf{p}\right) \tag{7″}$$

The problems (5) and (7′) describe essentially the same quantity viewed from two different perspectives: the Lagrangian and the Eulerian.[22] Indeed, taking into account the incompressibility of the flow, we can write the function (7) as

$$\Phi_t(\mathbf{r}, q, \mathbf{p} \mid \xi) = \delta(\xi(\mathbf{r}, t) - \xi)\, \Phi_{t,\mathbf{r}}(q, \mathbf{p})$$

where function $\xi(\mathbf{r}, t)$ inverts $\mathbf{r} = \mathbf{r}(t \mid \xi)$, i.e., restores the Lagrangian marker ξ from $(\mathbf{r}, t)$. Subsequent integration over the marker ξ yields the essential relation between the Eulerian and Lagrangian densities:

$$\Phi_{t,\mathbf{r}}(q, \mathbf{p}) = \int d\xi\, \Phi_t(\mathbf{r}, q, \mathbf{p} \mid \xi) \tag{8}$$

Since parameter ξ enters into (7′) only through the initial condition (7″), clearly, the equations for the Eulerian and the Lagrangian densities should coincide.

Let us also observe that the variable q in Eqs. (5) and (7′) enters only through the initial conditions. For that reason, multiplying those by q^n and integrating over q and $\mathbf{p}$ we get a dynamic evolution equation for moments q^n that coincides with (5) and (7′). The latter property is connected with the conservation of q along characteristics, and $q(\mathbf{r}, t) = q_0(\xi(\mathbf{r}, t))$.

To recapitulate, the quantities of interest to us, the Eulerian $q^n(\mathbf{r}, t)$ and $\Phi_{t,\mathbf{r}}(q, \mathbf{p})$, obey the same dynamic equations as the corresponding Lagrangian probability densities $\Phi_t(\mathbf{r} \mid \xi)$ and $\Phi_t(\mathbf{r}, q, \mathbf{p} \mid \xi)$.

In a similar manner we can consider a system of two particles

$$\begin{aligned}
&\frac{d\mathbf{r}_1(t)}{dt} = \mathbf{V}(\mathbf{r}_1, t), && \mathbf{r}_1(0) = \xi_1\\
&\frac{dp_{1i}(t)}{dt} = -\frac{\partial V_k(\mathbf{r}_1, t)}{\partial r_{1i}}\, p_{1k}(t), && p_{1i}(0) = \frac{\partial q_0(\xi_1)}{\partial \xi_{1i}}\\
&\frac{d\mathbf{r}_2(t)}{dt} = \mathbf{V}(\mathbf{r}_2, t), && \mathbf{r}_2(0) = \xi_2\\
&\frac{dp_{2i}(t)}{dt} = -\frac{\partial V_k(\mathbf{r}_2, t)}{\partial r_{2i}}\, p_{2k}(t), && p_{2i}(0) = \frac{\partial q_0(\xi_2)}{\partial \xi_{2i}}
\end{aligned} \tag{9}$$

The corresponding two-point Lagrangian density will then be described by the same equation as the two-point Eulerian density (4′). All (Eulerian and Lagrangian) densities above obey stochastic partial differential equations with the randomness introduced through the velocity fluctuations $\mathbf{F}$. Their ensemble averaging yields the evolution of the probability densities

$$P_{t,\mathbf{r}}(q, \mathbf{p}) = \langle \Phi_{t,\mathbf{r}}(q, \mathbf{p}) \rangle_{\mathbf{F}}, \qquad P_t(\mathbf{r}, q, \mathbf{p} \mid \xi) = \langle \Phi_t(\mathbf{r}, q, \mathbf{p} \mid \xi) \rangle_{\mathbf{F}} \quad (10)$$

in both the Eulerian, and the Lagrangian descriptions.

3. STATISTICAL AVERAGING

Here we shall implement the averaging procedure of Section 2 in a number of cases. For instance, averaging Eqs. (1) over the $\mathbf{F}$-ensemble yields an evolution of the mean field, where random velocities $\mathbf{F}$ are coupled to random solution $q = q[\mathbf{F}]$, itself a functional of $\mathbf{F}$, through the fluctuation term

$$\left\langle \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{r}} q \right\rangle \quad (11)$$

Here $\langle .,. \rangle$ means averaging over the space-time ensemble. So, to get the effective mean field evolution one needs to decouple the cross-correlation term (11). The decoupling methods strongly depend on the nature of the random field $\mathbf{F}$.

In the Gaussian case decoupling exploits the so-called *Furutsu–Novikov* formula.[23,24] Namely, given a zero-mean Gaussian random vector field $\mathbf{F} = (F_i)$, any functional $R[\mathbf{F}]$ satisfies

$$\langle F_i(\mathbf{r}, t)\, R[\mathbf{F}] \rangle = \int d\mathbf{r}' \int dt' \, \langle F_i(\mathbf{r}, t)\, F_j(\mathbf{r}', t') \rangle \left\langle \frac{\delta R[\mathbf{F}]}{\delta F_j(\mathbf{r}', t')} \right\rangle \quad (12)$$

So, the cross-term decouples into a superposition of products of the correlation coefficients and the mean variational derivatives of R. Applying the Furutsu–Novikov formula (12) to the cross-term (11) of Eq. (1), we get

$$\left(\frac{\partial}{\partial t} + \mathbf{v}(\mathbf{r}, t) \cdot \frac{\partial}{\partial \mathbf{r}} \right) \langle q(\mathbf{r}, t) \rangle + \int dt' \int d\mathbf{r}' \, B_{ij}(\mathbf{r}, t; \mathbf{r}', t') \frac{\partial}{\partial r_i} \left\langle \frac{\delta q(\mathbf{r}, t)}{\delta F_j(\mathbf{r}', t')} \right\rangle = \kappa \frac{\partial^2}{\partial \mathbf{r}^2} \langle q(\mathbf{r}, t) \rangle \quad (13)$$

where $B_{ij}(\mathbf{r}, t; \mathbf{r}', t') = \langle F_i(\mathbf{r}, t)\, F_j(\mathbf{r}', t') \rangle$ is the space-time correlation of $\mathbf{F}$. Although Eq. (13) is exact for any zero-mean Gaussian field $\mathbf{F}$, it is not closed, since the evolution of the mean field is coupled to the mean variational derivative with respect to **F**. The variational derivative $\delta q/\delta\mathbf{F}$ itself solves a stochastic differential equation

$$\frac{\partial}{\partial t}\left(\frac{\delta q(\mathbf{r}, t)}{\delta F_j(\mathbf{r}', t')}\right)+[\mathbf{v}(\mathbf{r}, t)+\mathbf{F}(\mathbf{r}, t)]\cdot\frac{\partial}{\partial\mathbf{r}}\left(\frac{\delta q(\mathbf{r}, t)}{\delta F_j(\mathbf{r}', t')}\right)$$
$$=\kappa\frac{\partial^2}{\partial\mathbf{r}^2}\left(\frac{\delta q(\mathbf{r}, t)}{\delta F_j(\mathbf{r}', t')}\right) \tag{14}$$

obtained by varying (1) in **F**, and satisfies the initial condition

$$\left.\frac{\delta q(\mathbf{r}, t)}{\delta F_j(\mathbf{r}', t')}\right|_{t\to t'+0}=-\delta(\mathbf{r}-\mathbf{r}')\frac{\partial}{\partial r_j}q(\mathbf{r}, t')$$

so $\delta q/\delta\mathbf{F}$ could be viewed as stochastic analog of the Green's function for problems of type (1). Taking the ensemble average of (14) and then applying the Furutsu–Novikov formula would produce higher order variational derivatives $\langle\delta^2 q/\delta F_i\,\delta F_j\rangle$ coupled together. Solution of such system of moment equations would require a suitable closure hypothesis that could be rigorously implemented only in certain cases. Two of these cases, namely the delta-correlated in time random fields $\mathbf{F}(\mathbf{r}, t)$ and the diffusion approximation for q, will be discussed below.

4. DELTA-CORRELATED RANDOM FIELD APPROXIMATION

This section discusses the case when the velocity field is delta-correlated in time. This assumption permits a great simplification of the general situation discussed above.

4.1. Mean Tracer Concentration and Its Correlation Function

In the delta-correlated approximation the random fluctuation field $\mathbf{F}(\mathbf{r}, t)$ is assumed to be zero-mean, Gaussian, with covariance structure

$$B_{ij}(\mathbf{r}, t; \mathbf{r}', t')=\langle F_i(\mathbf{r}, t)\,F_j(\mathbf{r}', t')\rangle=2B_{ij}^{\text{eff}}(\mathbf{r}; \mathbf{r}', t)\,\delta(t-t') \tag{15}$$

where

$$B_{ij}^{\text{eff}}(\mathbf{r}, \mathbf{r}', t)=\tfrac{1}{2}\int_{-\infty}^{\infty}dt'\,B_{ij}(\mathbf{r}, t; \mathbf{r}', t')$$

In this case, the integral term in (13) could be expressed through $\delta q(\mathbf{r}; t)/\delta F_j(\mathbf{r}'; t')$ at $t = t'$, i.e., through the initial condition of (14). As a result we get a closed-form differential equation for the mean

$$\left(\frac{\partial}{\partial t} + \mathbf{v}(r, t)\frac{\partial}{\partial \mathbf{r}}\right)\langle q(\mathbf{r}, t)\rangle = \frac{\partial}{\partial r_i}\left[B_{ij}^{\text{eff}}(\mathbf{r}; \mathbf{r}, t)\frac{\partial}{\partial r_j} + \kappa\frac{\partial}{\partial r_i}\right]\langle q(\mathbf{r}, t)\rangle$$

So, the spatial variance $B^{\text{eff}}(\mathbf{r}; \mathbf{r}, t)$ of $\mathbf{F}$ becomes the effective diffusion coefficient for the mean concentration.

If the velocity fluctuation field $\mathbf{F}(\mathbf{r}, t)$ is homogeneous and isotropic in space and stationary in time, then the effective correlation coefficients depend just on $|\mathbf{r} - \mathbf{r}'|$, i.e.,

$$B_{ij}^{\text{eff}}(\mathbf{r}; \mathbf{r}', t) = B_{ij}^{\text{eff}}(|\mathbf{r} - \mathbf{r}'|)$$

$$B_{ij}^{\text{eff}}(\mathbf{r} - \mathbf{r}') = \tfrac{1}{2}\int_{-\infty}^{\infty} dt'\, B_{ij}(\mathbf{r} - \mathbf{r}', t - t')$$

and their value at O is a scalar matrix with coefficient D_1:

$$B_{ij}^{\text{eff}}(0) = \delta_{ij} D_1, \qquad D_1 = \frac{1}{N} B_{ii}^{\text{eff}}(0) \tag{15'}$$

Here N ($=2$ or 3) denotes the dimension of space. Hence, we get

$$\left(\frac{\partial}{\partial t} + \mathbf{v}(r, t)\frac{\partial}{\partial \mathbf{r}}\right)\langle q(\mathbf{r}, t)\rangle = (D_1 + \kappa)\frac{\partial^2}{\partial \mathbf{r}^2}\langle q(\mathbf{r}, t)\rangle \tag{16}$$

In the particular case of zero mean flow $\mathbf{v}(\mathbf{r}, t) = 0$ and the initial tracer concentration $q_0(\mathbf{r})$ itself a homogeneous random field, the random solution $q(\mathbf{r}, t)$ will also be homogeneous and isotropic. Hence,

$$\langle q(\mathbf{r}, t)\rangle = q_0$$

Similarly, in this case, for the correlation function one obtains equation

$$\Gamma(\mathbf{r}, t) = \langle q(\mathbf{r}_1, t)\, q(\mathbf{r}_2, t)\rangle_{\mathbf{F}}, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

one obtains the equation

$$\frac{\partial}{\partial t}\Gamma(\mathbf{r}, t) = 2\kappa\frac{\partial^2}{\partial \mathbf{r}^2}\Gamma(\mathbf{r}, t) + 2\frac{\partial^2}{\partial r_i\, \partial r_j} D_{ij}(\mathbf{r})\,\Gamma(\mathbf{r}, t) \tag{17}$$

where

$$D_{ij}(\mathbf{r}) = B_{ij}^{\text{eff}}(0) - B_{ij}^{\text{eff}}(\mathbf{r}) \tag{17'}$$

is the matrix-valued structure function of field $\mathbf{F}$.

Let us remark that Eqs. (16) and (17) have the Fokker–Planck form for the one-particle and two-particle probability densities of the Langrangian coordinates. Furthermore, the Lagrangian relation (8) yields a Markov process. Additionally, Eq. (17) describes the relative diffusion of two particles. For sufficiently small initial distances between two particles ($r_0 \ll l_0$, where l_0 is the spatial radius of correlation of the fluctuation field $\mathbf{F}$) the function $D_{ij}(\mathbf{r})$ can be expanded in a Taylor series and in the first approximation

$$D_{ij}(\mathbf{r}) = \frac{1}{2} \frac{\partial^2 B_{ij}^{\text{eff}}(\mathbf{r})}{\partial r_k \, \partial r_l}\bigg|_{\mathbf{r}=0} r_k r_l \tag{18}$$

Now let us introduce the spectral density of the energy of the flow by the formula

$$B_{ij}^{\text{eff}}(\mathbf{r}) = \int d\mathbf{k} \; E(k) \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) e^{i\mathbf{k}\mathbf{r}} \tag{19}$$

Then

$$-\frac{\partial^2 B_{ij}^{\text{eff}}(\mathbf{r})}{\partial r_k \partial r_l}\bigg|_{\mathbf{r}=0} = D_2\{(N+1)\,\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}\} \tag{20}$$

where

$$D_2 = \frac{1}{N(N+2)} \int d\mathbf{k} \; \mathbf{k}^2 E(k) \tag{21}$$

Note that the quantity D_1 introduced earlier is also determined by the spectral density $E(k)$ via the equality

$$D_1 = \frac{N-1}{N} \int d\mathbf{k} \; E(k) \tag{21'}$$

In this case the diffusion tensor (18) simplifies and can be written in the form

$$D_{ij}(\mathbf{r}) = \tfrac{1}{2} D_2\{(N+1)\,\mathbf{r}^2\delta_{ij} - 2r_i r_j\} \tag{22}$$

Substituting (22) into (16), multiplying both sides of the obtained equation by $\mathbf{r}^2$, and integrating over $\mathbf{r}$, we obtain the equation

$$\frac{d}{dt}\langle \mathbf{r}^2(t)\rangle = 4\kappa N + 2(N+2)(N-1)\, D_2 \langle \mathbf{r}^2(t)\rangle \tag{23}$$

for variance $\langle \mathbf{r}^2 \rangle$ [the mean $\langle \mathbf{r}(t) \rangle$ is conserved]. Its solution has the structure

$$\langle \mathbf{r}^2 \rangle = r_0^2 e^{2(N+2)(N-1) D_2 t} + \frac{2\kappa N}{(N+2)(N-1) D_2} \{ e^{2(N+2)(N-1) D_2 t} - 1 \} \quad (24)$$

It is clear from (24) that under the condition

$$\kappa \ll D_2 r_0^2 \quad (25)$$

the effects of molecular diffusion on a particle are not significant and the last term in (24) can be omitted. In this case the solution becomes an exponentially growing function in time

$$\langle \mathbf{r}^2 \rangle = r_0^2 e^{2(N+2)(N-1) D_2 t} \quad (24')$$

Expression (24′) is valid whenever expansion (18) is, that is for the time range

$$D_2 t \ll \frac{1}{(N+2)(N-1)} \ln \frac{l_0}{r_0} \quad (26)$$

Note that the influence of the molecular diffusion for the above one-particle probability density, according to (16), can be neglected if the condition

$$\kappa \ll D_1 \quad (25')$$

is satisfied. Approximation (18) is, however, not valid for turbulent fluid flow,[1] for which the structure function cannot be expanded into a Taylor series.

As mentioned in the introduction, the mean value $\langle q(\mathbf{r}, t) \rangle$ and the correlation function $\Gamma(\mathbf{r}, t)$ characterize the spatiotemporal scales of the global passive tracer domain in the sense of statistical topography. At the same time they hide the detailed dynamics inside this domain. Clearly, the molecular diffusion coefficient has little influence on these scales and conditions (25) and (25′) are not very restrictive in the physical sense.

So far, we have considered the mean concentration of the tracer and its correlation function which are described in closed form due to the linearity of the basic equation (1). If one considers higher moments of the tracer concentration described by Eq. (1′) then one does not get a closed-form description. Indeed, averaging (1′) over the **F**-ensemble gives

$$\left(\frac{\partial}{\partial t} + \mathbf{v}(\mathbf{r}, t) \frac{\partial}{\partial \mathbf{r}} \right) \langle q^n(\mathbf{r}, t) \rangle$$

$$= (D_1 + \kappa) \frac{\partial^2}{\partial \mathbf{r}^2} \langle q^n(\mathbf{r}, t) \rangle - \kappa n(n-1) \langle q^{n-2}(\mathbf{r}, t)\, \mathbf{p}^2(\mathbf{r}, t) \rangle \quad (27)$$

whose right-hand side contains an unknown covariance of the concentration field and its spatial gradient. In order to understand better the structure of the tracer gradient field one can, in the first approximation, neglect effects of the molecular diffusion, i.e., consider the stochastic system (2)–(3). That will be done in the next section.

4.2. Fine Structure of Passive Tracer Fluctuations in Random Velocity Fields

In this subsection we look at subtler characteristics of tracer fluctuations than the means and correlations considered in Subsection 4.1, like the joint probability density of the tracer concentration and its gradient. To this end we average the Liouville equation (5) over the ensemble of realizations of fluctuation field $\mathbf{F}$ and use a version of the Furutsu–Novikov formula

$$\frac{\delta}{\delta u_j(\mathbf{r}', t-0)} \Phi_{t,\mathbf{r}}(q, \mathbf{p}) = \left\{ -\delta(\mathbf{r}-r') \frac{\partial}{\partial r_j} + \frac{\partial \delta(\mathbf{r}-r')}{\partial r_i} \frac{\partial}{\partial p_i} p_j \right\} \Phi_{t,\mathbf{r}}(q, \mathbf{p}) \quad (5')$$

for the variational derivative obtained from (5). As the result we get for the one-point joint probability density of fields $q(\mathbf{r}, t)$ and $\mathbf{p}(\mathbf{r}, t)$ the Fokker–Planck equation

$$\begin{aligned} &\left\{ \frac{\partial}{\partial t} + \mathbf{v}(\mathbf{r}, t) \frac{\partial}{\partial \mathbf{r}} - \frac{\partial \mathbf{p} \mathbf{v}(\mathbf{r}, t)}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{p}} \right\} P_{t,\mathbf{r}}(q, \mathbf{p}) \\ &\quad = D_1 \frac{\partial^2}{\partial \mathbf{r}^2} P_{t,\mathbf{r}}(q, \mathbf{p}) + D_2 \left\{ (N+1) \frac{\partial^2}{\partial \mathbf{p}^2} \mathbf{p}^2 + 2 \frac{\partial}{\partial \mathbf{p}} \mathbf{p} - 2 \left(\frac{\partial}{\partial \mathbf{p}} \right)^2 \right\} P_{t,\mathbf{r}}(q, \mathbf{p}) \end{aligned} \quad (28)$$

with initial condition

$$P_{0,\mathbf{r}}(q, \mathbf{p}) = \delta(q_0(r) - q)\, \delta\left(\frac{\partial}{\partial \mathbf{r}} q_0(\mathbf{r}) - \mathbf{p} \right) \quad (28')$$

Constants D_1 and D_2 introduced in (21) and (21′) become the new diffusion coefficients for the Fokker–Planck equation in the $\mathbf{r}$ and $\mathbf{p}$ spaces, respectively.

Equation (28) can be written in an operator form

$$\frac{\partial}{\partial t} P_{t,\mathbf{r}}(q, \mathbf{p}) = \hat{L}(\mathbf{r}, t)\, P_{t,\mathbf{r}}(q, \mathbf{p}) + \hat{M}(\mathbf{r}, \mathbf{p}, t)\, P_{t,\mathbf{r}}(q, \mathbf{p}) \quad (29)$$

where operators $\hat{L}$ and $\hat{M}$ are defined by

$$\hat{L}(\mathbf{r}, t) = -\mathbf{v}(\mathbf{r}, t)\frac{\partial}{\partial \mathbf{r}} + D_1 \frac{\partial^2}{\partial \mathbf{r}^2}$$

$$\hat{M}(\mathbf{r}, \mathbf{p}, t) = \frac{\partial \mathbf{p}\mathbf{v}(\mathbf{r}, t)}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}} + D_2\left\{(N+1)\frac{\partial^2}{\partial \mathbf{p}^2}\mathbf{p}^2 - 2\frac{\partial}{\partial \mathbf{p}}\mathbf{p} - 2\left(\frac{\partial}{\partial \mathbf{p}}\mathbf{p}\right)^2\right\} \tag{29'}$$

As was discussed earlier, operator $\hat{L}(\mathbf{r}, t)$ defines the spatial diffusion of the Lagrangian particle, while $\hat{M}(\mathbf{r}, \mathbf{p}, t)$ defines the diffusion of the tracer gradient and the correlation of gradient with the position vector. In the simplest case of zero mean flow ($\mathbf{v} = 0$), or in the case of shear flow with constant gradient, operators $\hat{L}$ and $\hat{M}$ commute, which reflects the statistical independence of diffusions in the position $\mathbf{r}$ space and gradient $\mathbf{p}$ space.

Also notice that in this case for spatially homogeneous and isotropic Gaussian velocity fluctuations, the corresponding diffusion operators are also isotropic. This fact will give us additional information on the fluctuations of the tracer gradient that we shall now outline.

Consider the case of the zero mean flow ($\mathbf{v} = 0$). Then the solution of Eq. (29) is obtained by averaging Eq. (8),

$$P_{t,\mathbf{r}}(q, \mathbf{p}) = \int d\xi\, P_t(\mathbf{r} \mid \xi)\, P_t(q, \mathbf{p} \mid \xi) \tag{30}$$

Here $P_t(\mathbf{r} \mid \xi)$ denotes the probability density of the particle Lagrangian coordinate given by

$$\frac{\partial}{\partial t} P_t(\mathbf{r} \mid \xi) = D_1 \frac{\partial^2}{\partial \mathbf{r}^2} P_t(\mathbf{r} \mid \xi), \qquad P_0(\mathbf{r} \mid \xi) = \delta(\mathbf{r} - \xi) \tag{31}$$

and $P_t(q, \mathbf{p} \mid \xi)$ is the joint probability density of q and ∇q, which satisfies

$$\frac{\partial}{\partial t} P_t(q, \mathbf{p} \mid \xi) = D_2\left\{(N+1)\frac{\partial^2}{\partial \mathbf{p}^2}\mathbf{p}^2 - 2\frac{\partial}{\partial \mathbf{p}}\mathbf{p} - 2\left(\frac{\partial}{\partial \mathbf{p}}\mathbf{p}\right)^2\right\} P_t(q, \mathbf{p} \mid \xi)$$
$$P_0(q, \mathbf{p} \mid \xi) = \delta q_0(\xi) - q)\, \delta(\mathbf{p}_0(\xi) - \mathbf{p}) \tag{32}$$

Specifically, in the 2D case the solution of Eq. (31) is

$$P_t(\mathbf{r} \mid \xi) = \exp\left[D_1 t \frac{\xi^2}{\partial \mathbf{r}^2}\right] \delta(\mathbf{r} - \xi) = (4\pi D_1 t)^{-1} \exp\left[-\frac{(\mathbf{r} - \xi)^2}{4 D_1 t}\right] \tag{31'}$$

It corresponds to the Brownian particle with parameters

$$\langle \mathbf{r}(t \mid \xi) \rangle = \xi, \qquad \sigma_{ij}^2(t) = 2D_1 \delta_{ij} t$$

From (32) we derive the following moment equation for gradient $\mathbf{p}(t \mid \xi)$:

$$\begin{aligned} \frac{\partial}{\partial t} \langle |\mathbf{p}(t \mid \xi)|^n \rangle &= D_2 n(N+n)(N-1) \langle |\mathbf{p}(t(\xi)|^n \rangle \\ \frac{\partial}{\partial t} \langle \mathbf{p} \, |\mathbf{p}(t \mid \xi)|^n \rangle &= D_2 n(N+n+2)(N-1) \langle \mathbf{p} \, |\mathbf{p}|^n \rangle \end{aligned} \tag{33}$$

and, in particular,

$$\frac{\partial}{\partial t} \langle \mathbf{p}(t \mid \xi) \rangle = 0, \qquad \text{i.e.,} \qquad \langle \mathbf{p}(t \mid \xi) \rangle = \mathbf{p}_0(\xi)$$

So the moment functions grow exponentially in time, with the exception of the conserved quantity $\langle \mathbf{p}(t \mid \xi) \rangle$. Also, for an arbitrary vector $\mathbf{a}$, the quantity $\langle |\mathbf{ap}(t \mid \xi)| \rangle$ is conserved, i.e.,

$$\langle |\mathbf{ap}(t \mid \xi)| \rangle = |\mathbf{ap}_0(\xi)| \tag{33'}$$

Notice that in the Eulerian representation, (30) implies that the exponential time growth of moments $\langle |\mathbf{p}(\mathbf{r}, t)|^n \rangle$ and $\langle \mathbf{p}(\mathbf{r}, t) \, |\mathbf{p}(\mathbf{r}, t)|^n \rangle$ is accompanied by their spatial dissipation with the diffusion rate D_1.

Equation (33) also implies that the normalized quantity $|\mathbf{p}(t \mid \xi)| / |\mathbf{p}_0(\xi)|$ has a log-normal probability distribution, i.e.,

$$\chi(t) = \ln \frac{|\mathbf{p}(t \mid \xi)|}{|\mathbf{p}_0(\xi)|}$$

is Gaussian with parameters

$$\langle \chi(t) \rangle = D_2 N(N-1)\, t, \qquad \sigma_\chi^2(t) = 2D_2(N-1)\, t \tag{34}$$

Properties of the log-normal distribution were studied in detail in ref. 25, where it was shown that the typical realization of process $|\mathbf{p}(t)|$ has an exponential growth

$$|\mathbf{p}(t \mid \xi)| \sim |\mathbf{p}_0(\xi)| \exp[D_2 N(N-1)\, t]$$

accompanied by large excursions relative to the above exponential curve. In addition, there exist several lower probabilistic estimates for the quantity $\chi(t)$. Note that this situation is fundamentally different from the one-dimensional problem (where the fluid flow is always compressible). There, the gradient conserves its sign and a typical realization of the gradient process is an exponentially decaying curve.[22, 26] It is worth mentioning that the log-normal distribution of the norm of the tracer gradient, first proposed in ref. 27, agrees well with experimental atmospheric data.[28, 29] This law was initially discovered theoretically in ref. 30, although without the equations obtained above.

We also get from (32) the evolution equation

$$\frac{\partial}{\partial t}\langle p_i(t\,|\,\xi)\, p_j(t\,|\,\xi)\rangle$$
$$= -4D_2\langle p_i(t\,|\,\xi)\, p_j(t\,|\,\xi)\rangle + 2D_2(N+1)\,\delta_{ij}\langle \mathbf{p}^2(t)\,|\,\xi\rangle \qquad (35)$$

for the covariance $\langle p_i(t\,|\,\xi)\, p_j(t\,|\,\xi)\rangle$. Clearly, cross-terms $i \neq j$ of the correlation of different components of gradient $p(t\,|\,\xi)$ converge rapidly (exponentially) to zero. So, for large time values $D_2 t \gg 1/4$ the vector $\mathbf{p}(t\,|\,\xi)$ undergoes full statistical isotropization independent of the initial conditions.

Here we have limited our attention to moments of the tracer gradient (33), (33′). Saichev and Woyczynski[26] concentrate on a geometric interpretation of these quantities and we will analyze them from that (statistical topographic) perspective below.

4.3. Geometric Interpretation of the Fine Structure (Statistical Topography)

In the previous sections we have obtained a series of general equations which in principle permit us to obtain information about the time evolution of one-point, two-point, etc., probability densities. The complete set of these equation, obviously, will give also the exhaustive description of the behavior of separate realizations of solutions of the initial stochastic equations. However, in practice, even the one-point probability densities for solutions of the stochastic equations can be calculated only in a few special cases. So in our case of mean flow it becomes impossible. Even in cases when this could be done and various statistical characteristics of solutions could be computed they would behave differently than individual realizations of the original stochastic system.

In this context an important question has to be addressed: how can we obtain information on geometric properties of individual realizations of random fields from partial information of their statistical characteristics? This question is especially pertinent for real physical systems like oceans and the atmosphere, where, generally speaking, one deals with concrete realizations rather than ensembles. The study of these problems is the subject matter of the *statistical topography* of random fields (see, e.g., refs. 16 and 17).

The structure of the spatial random field $q(\mathbf{r}, t)$ of the passive tracer is highly chaotic, its individual realizations constantly change their shape, and are characterized by "sharp peaks", saddles, ridges, etc. Averaging clearly smoothes out all special features of individual realizations. The level curves of such a "rough system" driven by stochastic flows also obey a stochastic time evolution determined by the equation $q(\mathbf{r}, t) = q$. The mean values of distributional indicator functions $\phi_{t,\mathbf{r}}(q) = \delta(q(\mathbf{r}, t) - q)$ of these level curves define the corresponding probability density. The function $\phi_{t,\mathbf{r}}(q)$ determines a surface S of constant values of, for example, concentration, temperature, etc., in the 3D space, and an analogous contour l in the 2D space.

In this subsection, as in ref. 26, we will restrict our attention to the case of two-dimensional fluid flows.

4.3.1. Level Curve Length Statistics. Consider the following auxiliary integral related to the function $\phi_{t,\mathbf{r}}(q)$:

$$A_n(t) = \int d\mathbf{r}\, |\mathbf{p}(\mathbf{r}, t)|^{n+1}\, \delta[q(\mathbf{r}, t) - q] = \oint |\mathbf{p}(\mathbf{r}, t)|^n\, dl \tag{36}$$

where $\mathbf{p}(\mathbf{r}, t) = \partial q(\mathbf{r}, t)/\partial \mathbf{r}$ is the spatial gradient of the random field $q(\mathbf{r}, t)$. These integrals are moments of density gradients integrated over contours.

In particular, for $n = 0$, formula (36) gives the length of the contour

$$A_0(t) = l(t) = \int d\mathbf{r}\, |\mathbf{p}(\mathbf{r}, t)|\, \delta[q(\mathbf{r}, t) - q] = \oint dl \tag{36$'$}$$

The mean tracer concentration gradient over the planar level sets is given by the contour integral

$$\mathbf{A}(t) = \int_S \nabla q(\mathbf{r}, t)\, dS = q \int d\mathbf{r}\, \mathbf{p}(\mathbf{r}, t)\, \delta[q(\mathbf{r}, t) - q]$$

$$= q \oint \frac{\mathbf{p}(\mathbf{r}, t)}{|\mathbf{p}(\mathbf{r}, t)|}\, dl \tag{36$''$}$$

Equation (36)–(36″) can be rewritten in terms of the distribution function $\Phi_{t,\mathbf{r}}(q, \mathbf{p})$ as follows:

$$A_n(t) = \int d\mathbf{r} \int d\mathbf{p}\, p^{n+1} \Phi_{t,\mathbf{r}}(q, \mathbf{p})$$
$$\mathbf{A}(t) = q \int d\mathbf{r} \int d\mathbf{p}\, \mathbf{p} \Phi_{t,\mathbf{r}}(q, \mathbf{p}) \tag{37}$$

Consequently, their averages are determined by the one-point probability density $P_{t,\mathbf{r}}(q, \mathbf{p})$ via

$$\langle A_n(t) \rangle = \int d\mathbf{r} \int d\mathbf{p}\, p^{n+1} P_{t,\mathbf{r}}(q, \mathbf{p})$$
$$\langle \mathbf{A}(t) \rangle = q \int d\mathbf{r} \int d\mathbf{p}\, \mathbf{p} P_{t,\mathbf{r}}(q, \mathbf{p}) \tag{37'}$$

Substituting (30) for $P_{t,\mathbf{r}}(q, \mathbf{p})$ and taking into account that $\int d\mathbf{r}\, P_t(\mathbf{r} \mid \xi) = 1$, we express the quantities of interest

$$\langle A_n(t) \rangle = \int d\xi \int d\mathbf{p}\, p^{n+1} P_t(q, \mathbf{p} \mid \xi)$$
$$\langle \mathbf{A}(t) \rangle = q \int d\xi \int d\mathbf{p}\, \mathbf{p} P_t(q, \mathbf{p} \mid \xi) \tag{37''}$$

in terms of Lagrangian probability density. In the 2D case, those satisfy Eq. (32),

$$\frac{\partial}{\partial t} P_t(q, \mathbf{p} \mid \xi) = D_2 \left\{ 3 \frac{\partial^2}{\partial \mathbf{p}^2} \mathbf{p}^2 - 2 \frac{\partial}{\partial \mathbf{p}} \mathbf{p} - 2 \left(\frac{\partial}{\partial \mathbf{p}} \mathbf{p} \right)^2 \right\} P_t(q, \mathbf{p} \mid \xi)$$
$$P_0(q, \mathbf{p} \mid \xi) = \delta(q_0(\xi) - q)\, \delta(\mathbf{p}_0(\xi) - \mathbf{p})$$
$$\mathbf{p}_0(\xi) = \frac{\partial q_0(\xi)}{\partial \xi}$$

Differentiating expression (37″) with respect to time and applying equality (32) with $N = 2$, we obtain differential equations for the mean values

$$\frac{d}{dt} \langle A_n(t) \rangle = (n+1)(n+3)\, D_2 \langle A_n(t) \rangle, \qquad \langle A_n(0) \rangle = A_n(0)$$

$$\frac{d}{dt} \langle \mathbf{A}(t) \rangle = 0, \qquad \langle \mathbf{A}(0) \rangle = \mathbf{A}(0)$$

Their solutions give functions exponentially growing in time

$$\langle lt)\rangle = l_0 e^{3D_2 t}, \qquad \langle A_n(t)\rangle = A_n(0)\, e^{(1+n)(n+3)\, D_2 t} \tag{38}$$

whereas the mean concentration gradient, averaged over the area, is conserved, i.e. $\langle \mathbf{A}(t)\rangle = \mathbf{A}(0)$.

The exponential growth of (38) indicates strong roughening of the tracer level curves with time, which leads to "fractal-like" structures. The situation is similar for the tracer density and its gradient. Examples of numerical modeling of this phenomenon are shown in Fig. 1 (for more details see ref. 18).

4.3.2. Area Statistics for Level Curves. Let us note the following expression for the area bounded by the level curve of the field $q(\mathbf{r}, t)$:

$$S(t, q) = \pm \tfrac{1}{2} \int d\mathbf{r}\, (\mathbf{r}\mathbf{p})\, \Phi_{t,\mathbf{r}}(q) \tag{39}$$

where the choice of the $\pm$ sign is determined by the value of $S(t, q)$ at $t = 0$, or by the type of monotonicity of q [in the case $q(\mathbf{r}, t)$ varies monotonically]. Thus, if the field $q(\mathbf{r}, t)$ is radial, i.e., $q(\mathbf{r}, t) = q(r, t)$, then

$$\mathbf{p}(\mathbf{r}, t) = \frac{\mathbf{r}}{r} \frac{\partial}{\partial r} q(r, t)$$

and the sign depends on the sign of the derivative $\partial q(r, t)/\partial r$. One can integrate (39) by parts to get the relation

$$\frac{\partial}{\partial q} S(t, q) = \pm \int d\mathbf{r}\, \Phi_{t,\mathbf{r}}(q) = \pm A_{-1}(t, q) \tag{40}$$

It is obvious that in the more general case

$$F_S(t; q) = \int_S F(t, \mathbf{r}; q(\mathbf{r}, t))\, d\mathbf{r}$$

integrated over the level set S bounded by the curve $q(\mathbf{r}, t) = q$ we have

$$\frac{\partial}{\partial q} F_S(t, q) = \pm \int d\mathbf{r}\, F(t, \mathbf{r}; q)\, \Phi_{t,\mathbf{r}}(q) \tag{40'}$$

In particular

$$\frac{\partial}{\partial q}\int_{S(t,q)} d\mathbf{r}\, F(q,(\mathbf{r},t)) = \pm F(q)\frac{\partial}{\partial q} S(t,q) \tag{40''}$$

and, consequently, the integral $\int_{S(t,q)} d\mathbf{r}\, F(q(\mathbf{r},t))$ is independent of time, because $S(t,q) = S(0,q)$ in view of the incompressibility of the flow.

4.3.3. Mean Number of Level Contours.

Here we consider the time evolution of the level contours

$$q(\mathbf{r},t) = q = \text{const} \tag{41}$$

Figure 2 provides a schematic illustration of how the initial connected constant concentration level contour evolves in time into several connected components.

The dynamics of q increases the complexity of level curves and leads to their fragmentation into disconnected contours. This process is partly described by statistics of the contour number $\mathcal{N}(t,q)$, which admits the following geometric estimate expressed in terms of q and ∇q:

$$\mathcal{N}(t,q) \leqslant \int_0^\infty dr \left|\frac{\partial q(\mathbf{r},t)}{\partial r}\right| \delta(q(\mathbf{r},t)-q) \tag{42}$$

The estimate is written in polar coordinates (r, ϕ). We look at each direction ϕ and count the number of intersections along the ray (r, ϕ) with the level set (41). The right-hand side of (43) assigns to each level curve the

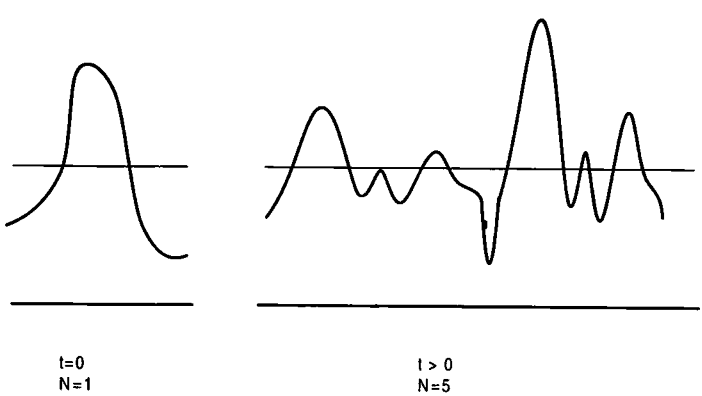

Fig. 2. Schematic evolution of a simple initial profile ($\mathcal{N}=1$) at $t=0$ into a more complicated topographic pattern ($\mathcal{N}=5$) at a later time.

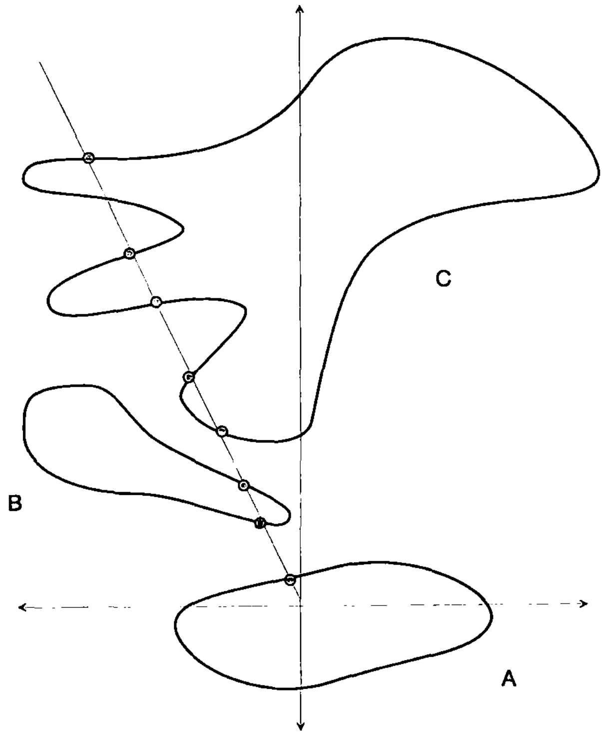

Fig. 3. Calculation of the estimate for the number of disconnected contours of a level curve. The number of intersections for contours A, B, and C is, respectively, 1, 2, and 6.

maximal number to its radial branches (see Fig. 3), and hence provides an obvious estimate of $\mathcal{N}$.

One could write an exact expression for $\mathcal{N}$ in terms of curvature κ, namely,

$$\mathcal{N}(t, q) = (1/2\pi) \int d\mathbf{r}\, \kappa(\mathbf{r}, t)\, |\nabla q(\mathbf{r}, t)|\, \delta(q(\mathbf{r}, t) - q)$$

However, κ involves a complicated expression in terms of first and second derivatives of q, that are too difficult to analyze statistically.

Taking into account the fact that

$$\frac{\partial}{\partial r} q(\mathbf{r}, t) = \frac{\mathbf{r}}{r} \nabla q(\mathbf{r}, t) \tag{43}$$

one can rewrite formula (42) in the form

$$\mathcal{N}(t,q) \leqslant \int_0^\infty dr \int d\mathbf{p} \frac{|\mathbf{rp}|}{r} \Phi_{t,\mathbf{r}}(q,\mathbf{p}) \tag{44}$$

If we average now inequality (44) over the ensemble of realizations, in view of (30)–(32) we obtain the following estimate for the average number of disconnected components of the level curve:

$$\langle \mathcal{N}(t,q)\rangle \leqslant \int_0^\infty dr \int d\mathbf{p} \int d\boldsymbol{\xi} \frac{|\mathbf{rp}|}{r} P_t(\mathbf{r}\,|\,\boldsymbol{\xi})\, P_t(\mathbf{p}\,|\,\boldsymbol{\xi})\, \delta(q_0(\boldsymbol{\xi})-q) \tag{45}$$

Since the quantity $\langle\, |\mathbf{rp}(t\,|\,\boldsymbol{\xi})|\rangle$ is conserved during time evolution [see, e.g., (33′)], the integration over $\mathbf{p}$ in (45) can be carried out to give

$$\langle \mathcal{N}(t,q)\rangle \leqslant \int_0^\infty dr \int d\boldsymbol{\xi} \frac{|\mathbf{rp}_0(\boldsymbol{\xi})|}{r} P_t(\mathbf{r}\,|\,\boldsymbol{\xi})\, \delta(q_0(\boldsymbol{\xi})-q) \tag{46}$$

Now let us assume that the initial distribution of the passive tracer is radial, i.e., $q_0(\boldsymbol{\xi}) \equiv q_0(\xi)$. Then,

$$\mathbf{p}_0(\boldsymbol{\xi}) = \frac{\partial q_0(\xi)}{\partial \xi} \cdot \frac{\boldsymbol{\xi}}{\xi}$$

and the inequality (46), taking into account (31′), can be rewritten in the form

$$\langle \mathcal{N}(\tau)\rangle \leqslant (4/\pi\tau) \int_0^{\pi/2} d\phi \int_0^\infty dr$$
$$\times \cos\phi\, e^{-(1+r^2)/\tau} \cosh(2r\cos\phi/\tau) \tag{47}$$

where the dimensionless time $\tau = 4D_1 t/r_0^2(q)$ has been introduced and where $r_0(q)$ is the radius of the initial level concentration curve. It follows from (47) that for $\tau \gg 1$ one has the asymptotic

$$\langle \mathcal{N}(\tau)\rangle = \frac{2}{\sqrt{\pi\tau}} \tag{48}$$

i.e., the mean number of connected components of the concentration level curve decreases in time according to a power law. Figure 4 shows the dependence of the estimate of $\langle \mathcal{N}(\tau)\rangle$ on the dimensionless time τ as well as the asymptotic expression (48). This dependence is totally determined by

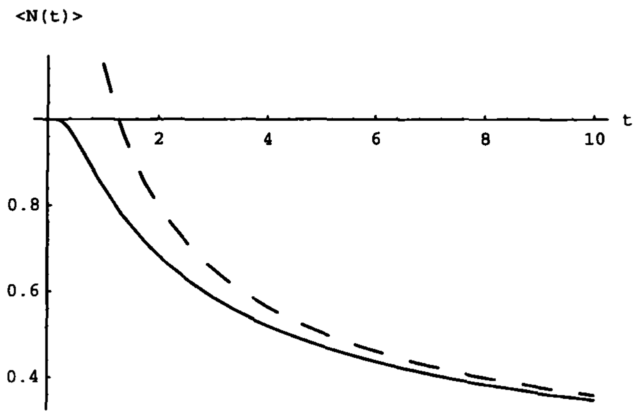

Fig. 4. Estimate of the mean number $\mathcal{N}(t, q)$ of connected component contours of level curves and its asymptotics (dashed line).

the diffusion coefficient D_1 and is independent of the fine structure of fluctuations of the tracer concentration gradient. Let us remark that the exact mean contour number $\langle N(t, q)\rangle$ would depend on the diffusion coefficient D_2 that describes the fine structure of the tracer field.

In this subsection, we have provided a detailed statistical analysis of tracer concentration in random velocity fields and, in the absence of a mean flow, the analysis of the tracer gradient. The presence of a mean flow (even a deterministic one) leads to steepening of the tracer gradient and the deformation of its level sets. The presence of even small fluctuations of the velocity field quickly accelerates these processes. We will illustrate this by the example of the simplest two-dimensional linear shear flow.

4.4. Linear Shear Flow

In this subsection we consider a linear shear mean flow

$$v_x = \alpha y, \qquad v_y = 0$$

and analyze the effect of the shear on the statistics of the tracer concentration.

The probability distribution $P_{t,\mathbf{r}}(q, \mathbf{p})$ is described by Eq. (28) for $N = 2$, which now takes the form

$$\frac{\partial}{\partial t} P_{t,\mathbf{r}}(q, \mathbf{p})$$

$$= \left\{ -\alpha y \frac{\partial}{\partial x} + D_1 \frac{\partial^2}{\partial \mathbf{r}^2} \right\} P_{t,\mathbf{r}}(q, \mathbf{p})$$

$$+ \left\{ \alpha p_1 \frac{\partial}{\partial p_2} + D_2 \left[3 \frac{\partial^2}{\partial \mathbf{p}^2} \mathbf{p}^2 - 2 \frac{\partial}{\partial \mathbf{p}} \mathbf{p} - 2 \left(\frac{\partial}{\partial \mathbf{p}} \mathbf{p} \right)^2 \right] \right\} P_{t,\mathbf{r}}(q, \mathbf{p}) \qquad (49)$$

where, we recall that $(p_1, p_2) = (p_x, p_y)$. As before, the effective spatial diffusion of particles will be described by the operator

$$\hat{L}(\mathbf{r}) = \alpha y \frac{\partial}{\partial x} + D_1 \frac{\partial^2}{\partial \mathbf{r}^2}$$

The corresponding probability distribution is Gaussian with parameters[31]

$$\langle x(t) \rangle = x_0 + \alpha y_0 t, \qquad \langle y(t) \rangle = y_0$$

$$\sigma_{xx}^2(t) = 2D_1 t(1 + \tfrac{1}{3}\alpha^2 t^2), \qquad \sigma_{yy}^2 = 2D_1 t, \qquad \sigma_{xy}^2 = \alpha D_1 t^2$$

The operator

$$\hat{M}(\mathbf{p}) = \alpha p_1 \frac{\partial}{\partial p_2} + D_2 \left[3 \frac{\partial^2}{\partial \mathbf{p}^2} \mathbf{p}^2 - 2 \frac{\partial}{\partial \mathbf{p}} \mathbf{p} - 2 \left(\frac{\partial}{\partial \mathbf{p}} \mathbf{p} \right)^2 \right]$$

describes the diffusion of the tracer concentration gradient and is now anisotropic. In this case the mean value of the vector $\mathbf{p}$ is not conserved; it gives a linear function of t,

$$\langle p_1(t) \rangle = p_1(0), \qquad \langle p_2(t) \rangle = p_2(0) - \alpha p_1(0)\, t$$

in the case of zero velocity fluctuations.

We consider second moments of the vector $\mathbf{p}$ and write the Lagrangian equations for them:

$$\frac{d}{dt} \langle \mathbf{p}^2 \rangle = 8D_2 \langle \mathbf{p}^2 \rangle - 2\alpha \langle p_1 p_2 \rangle$$

$$\frac{d}{dt} \langle p_1 p_2 \rangle = -4D_2 \langle p_1 p_2 \rangle - \alpha \langle p_1^2 \rangle \qquad (50)$$

$$\frac{d}{dt} \langle p_1^2 \rangle = -4D_2 \langle p_1^2 \rangle + 6D_2 \langle \mathbf{p}^2 \rangle$$

The linear ordinary differential system (50) has characteristic exponents λ that obey

$$(\lambda + 4D_2)^2 (\lambda - 8D_2) = 12\alpha^2 D_2 \tag{51}$$

The roots of the characteristic equation (51), depend, essentially, on α/D_2.

For small $\alpha/D_2 \ll 1$, these roots are, approximately,

$$\lambda_1 = 8D_2 + \frac{1}{12}\frac{\alpha^2}{D_2}, \qquad \lambda_2 = -4D_2 + i\,|\alpha|, \qquad \lambda_3 = -4D_2 - i\,|\alpha| \tag{52}$$

Hence, in the time range $D_2 t \gg 1/4$, the solution of the problem is completely controlled by random factors. This means that random velocity fluctuations become quickly dominant in problems with weak mean-field gradients.

In the case of large $\alpha/D_2 \gg 1$, the characteristic equation (51) has approximate roots

$$\lambda_1 = (12\alpha^2 D_2)^{1/3}, \qquad \lambda_2 = (12\alpha^2 D_2)^{1/3}\, e^{i(2/3)\pi}, \qquad \lambda_3 = (12\alpha^2 D_2)^{1/3}\, e^{-i(2/3)\pi} \tag{53}$$

Since the real parts of λ_2 and λ_3 are negative, for $(12\alpha^2 D_2)^{1/3}\, t \gg 1$, solutions are asymptotic to

$$\langle \mathbf{p}^2(t) \rangle \sim \exp\{(12\alpha^2 D_2)^{1/3}\, t\} \tag{54}$$

so even small velocity fluctuations have significant effect on the second moment in sufficiently strong mean gradient flows.

4.5. Effects of Molecular Diffusion

We shall start with an observation made earlier: as time passes, the tracer concentration field acquires a more chaotic structure and its spatial gradient steepens. In addition, fine-scale structures are created. This tendency would be checked at the level of molecular diffusion, so the dynamical picture would be valid only for a limited time interval. Our goal is to estimate the length of this time interval. To that end we shall utilize the exact equation (27) in the absence of mean flow:

$$\frac{\partial}{\partial t}\langle q^n \mathbf{r}, t)\rangle = (D_1 + \kappa)\frac{\partial^2}{\partial \mathbf{r}^2}\langle q^n(\mathbf{r}, t)\rangle - \kappa n(n-1)\langle q^{n-2}(\mathbf{r}, t)\,\mathbf{p}^2(\mathbf{r}, t)\rangle \tag{55}$$

For small $\kappa \ll D_1$, the solution of (55) can be written in the form

$$\langle q^n(\mathbf{r}, t)\rangle = \exp\left(tD_1 \frac{\partial^2}{\partial \mathbf{r}^2}\right) q_0^n(\mathbf{r}) - \kappa n(n-1) \int_0^t d\tau \exp\left[(t-\tau) D_1 \frac{\partial^2}{\partial \mathbf{r}^2}\right] \times \langle q^{n-2}(\mathbf{r}, \tau) \mathbf{p}^2(\mathbf{r}, \tau)\rangle \tag{56}$$

To evaluate the last term in (56), we exploit Eq. (29) with zero mean flow. Then we get

$$\frac{\partial}{\partial t} \langle q^{n-2}(\mathbf{r}, t) \mathbf{p}^2(\mathbf{r}, t)\rangle = D_1 \frac{\partial^2}{\partial \mathbf{r}^2} \langle q^{n-2}(\mathbf{r}, t) \mathbf{p}^2(r, t)\rangle + 2D_2(N+2)(N-1)\langle Q^{n-2}(\mathbf{r}, t) \mathbf{p}^2(\mathbf{r}, t)\rangle$$

for $\langle q^{n-2}(\mathbf{r}, t) \mathbf{p}^2(\mathbf{r}, t)\rangle$. Its solution is

$$\langle q^{n-2}(\mathbf{r}, t) \mathbf{p}^2(\mathbf{r}, t)\rangle = \exp\left[2D_2(N+2)(N-1)\, t + D_1 t \frac{\partial^2}{\partial \mathbf{r}^2}\right] q_0^{n-2}(\mathbf{r})\, \mathbf{p}_0^2(\mathbf{r}) \tag{57}$$

Substituting (57) into (56) and carrying out integration with respect to τ, we get

$$\langle q^n(\mathbf{r}, t)\rangle = \exp\left(tD_1 \frac{\partial}{\partial \mathbf{r}^2}\right)\left\{q_0^n(\mathbf{r}) - \kappa \frac{n(n-1)}{2D_2(N+2)(N-1)} \times \{\exp[2D_2(N+2)(N-1)\, t] - 1\}\, q_0^{n-2}(\mathbf{r})\, p_0^2(\mathbf{r})\right\} \tag{58}$$

Formula (58) shows that the molecular diffusion becomes insignificant under two conditions. One of them limits the initial characteristic size of tracer from below:

$$2(N+2)(N-1)\, D_2 r_0^2 \gg \kappa n(n-1) \tag{59}$$

Here, r_0 is the characteristic size of initial tracer concentration $q_0(\mathbf{r})$ [cf. (25)]. The other condition limits the time range by

$$D_2 t \ll \frac{1}{2(N+2)(N-1)} \ln \frac{D_2 r_0^2}{\kappa n^2} \tag{60}$$

Notice that the time domain (60) decreases with the growth of the power n.

Note that, the case of linear initial tracer concentration

$$q_0(\mathbf{r}) = \mathbf{G}\mathbf{r}, \qquad \mathbf{p}_0(\mathbf{r}) = \mathbf{G} = (G_1, G_2)$$

permits more complete analysis.[32]

This problem has recently attracted considerable attention from both the theoretical and experimental side.[33–39] These papers used numerical modeling and phenomenological models to analyze the behavior of the stationary ($t \to \infty$) probability density of the tracer gradient. They observed, among others, the appearance of distributions with "slowly decaying tails" of exponential type.

Representing concentration $q(\mathbf{r}, t)$ in the form

$$q(\mathbf{r}, t) = \mathbf{G}\mathbf{r} + \tilde{q}(\mathbf{r}, t)$$

we obtain for the fluctuating component $\tilde{q}$

$$\left(\frac{\partial}{\partial t} + \mathbf{F}\frac{\partial}{\partial \mathbf{r}}\right)\tilde{q}(\mathbf{r}, t) = -\mathbf{G}\mathbf{F}(\mathbf{r}, t) + \kappa\frac{\partial^2}{\partial \mathbf{r}^2}\tilde{q}(\mathbf{r}, t), \qquad \tilde{q}(\mathbf{r}, 0) = 0 \tag{61}$$

Statistical spatial homogeneity of the field $\tilde{q}(\mathbf{r}, t)$ makes analysis of Eq. (61) simpler than the original problem (1).

In particular, the stationary value of the second moment of the concentration gradient for the Gaussian, delta-correlated in time random field $\mathbf{F}(\mathbf{r}, t)$ is

$$\langle \tilde{\mathbf{p}}^2(\mathbf{r}, t)\rangle = \langle [\tilde{q}(\mathbf{r}, t)]^2\rangle = D_1 \mathbf{G}^2/\kappa \tag{62}$$

whereas its mean and the variance become

$$\langle \tilde{q}(\mathbf{r}, t)\rangle = 0, \qquad \langle \tilde{q}^2(\mathbf{r}, t)\rangle = 2D_1 \mathbf{G}^2 \int_0^t d\tau\, f(\tau) \tag{63}$$

with

$$f(t) = 1 - \frac{\kappa}{D_1 \tilde{G}^2}\langle \tilde{\mathbf{p}}^2(\mathbf{r}, t)\rangle$$

At initial stages of time evolution the statistical characteristics of the gradient do not depend on the molecular diffusion κ in view of the formula (33), and

$$\langle \mathbf{p}(\mathbf{r}, t)\rangle = \mathbf{G}, \qquad \langle |\tilde{\mathbf{p}}(\mathbf{r}, t)|^2\rangle = \mathbf{G}^2\{e^{2D_2(N+2)(N-1)t} - 1\} \tag{64}$$

The solution (64) makes it possible to estimate the time T_0 when the quantity $\langle \tilde{\mathbf{p}}^2(\mathbf{r}, t)\rangle$ attains its stationary value $\langle \tilde{\mathbf{p}}^2\rangle$, described by Eq. (62). Namely,

$$T_0 \sim \frac{1}{2D - 2(N+2)(N-1)} \ln \frac{D_1 + \kappa}{\kappa} \tag{65}$$

Hence, $\int_0^\infty dt\, f(t) \sim T_0$ and, by (63), we get the stationary variance of $\tilde{q}(\mathbf{r}, t)$,

$$\lim_{t \to \infty} \langle \tilde{q}^2(\mathbf{r}, t)\rangle \sim \frac{1}{(N+2)(N-1)} \frac{D_1}{D_2} \mathbf{G}^2 \ln \frac{D_1 + \kappa}{\kappa} \tag{66}$$

Taking into account that

$$D_1 \sim \sigma_u^2 t_0, \qquad D_1/D_2 \sim l_0^2$$

with velocity variance σ_u^2, and temporal and spatial correlation radii t_0 and l_0, it follows from (65)–(66) that the time T_0 can not be too large, due to its logarithmic dependence on the parameter κ. Furthermore,

$$\langle \tilde{q}^2\rangle \sim \mathbf{G}^2 l_0^2 \ln \frac{\sigma_u^2 t_0}{\kappa}, \qquad \kappa \ll \sigma_u^2 \tau_0$$

5. DIFFUSION APPROXIMATION

In the previous section, we provided a detailed statistical topography analysis of passive tracer transport in the case of the delta-correlated velocity fluctuations $\mathbf{F}$. The condition of applicability of such an approximation is that the temporal correlation radius t_0 of $\mathbf{F}$ is much smaller than any other temporal scale arising in the problem.[22]

For problems considered in Section 4.1 and 4.2, in the absence of mean flow, the temporal scales connected with the statistics of the fluctuating velocity component include

$$r_0^2/D_1, \qquad l_0^2/D_1, \qquad 1/D_2$$

Here, r_0 is the characteristic dimension of the initial tracer concentration $q_0(\mathbf{r})$, l_0 is the spatial correlation radius of the velocity field, and D_1, D_2 are diffusion coefficients introduced in (21)–(21′). Other temporal scales, related to the coefficient of molecular diffusion, are

$$r_0^2/\kappa, \qquad l_0^2/\kappa$$

In the presence of the mean flow, new temporal scales appear. Thus, for the mean flow considered in Section 4.4, such an additional scale is the quantity $1/\alpha$—the reciprocal shear gradient.

In the diffusion approximation we assume that velocity fluctuations $\mathbf{F}$ do not affect the dynamics of q on scales of the order of t_0. Hence, the dynamics of the passive tracer at these temporal scales can be approximately described by

$$\left(\frac{\partial}{\partial t}+\mathbf{v}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\frac{\delta q(\mathbf{r},t)}{\delta F_j(\mathbf{r}',t)}=\kappa\frac{\partial^2}{\partial \mathbf{r}^2}\frac{\delta q(\mathbf{r},t)}{\delta F_j(\mathbf{r}',t')}$$

$$\frac{\delta q(\mathbf{r},t')}{\delta F_j(\mathbf{r}',t')}=-\delta(\mathbf{r}-\mathbf{r}')\frac{\partial}{\partial r_j}q(\mathbf{r},t')$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}(\mathbf{r},t)\cdot\frac{\partial}{\partial \mathbf{r}}\right)q(\mathbf{r},t)=\kappa\frac{\partial^2}{\partial \mathbf{r}^2}q(\mathbf{r},t)$$

$$q(\mathbf{r},t)|_{t=t'}=q(\mathbf{r},t')$$

These equations are deterministic and randomness enters here only through the initial condition. In this approximation, the field $q(\mathbf{r},t)$ is Markovian at large time scales $t \gg t_0$.[22]

Let us also remark that the diffusion approximation becomes exact for the linear equation with an additive noise. Here, the variational derivative coincides with the Green's function of the deterministic equation.

Examples of applications of the diffusion approximation and specific estimates for the delta-correlated field $\mathbf{F}(\mathbf{r},t)$ are provided in ref. 22. It is used to analyze two-dimensional particle diffusion by a Gaussian incompressible velocity field with parallel mean flow. In ref. 32 a similar analysis is applied to fluctuations of a passive tracer with mean concentration gradient.

Here, we shall consider the practically important problem of a diffusing passive tracer that undergoes a sedimentation in an isotropic random velocity fields.[5]

Dispersion of particles affected by gravity and buoyancy forces plays important role in climatological and ecological models. Examples include grained dust emitted by industrial plants or sites of ecological disasters such as forrest fires or volcanic eruptions. The velocity $\mathbf{v}$ of sedimentation or buoyancy is directed along the vertical direction and is determined by the balance of buoyant forces and the viscous friction forces. If the particle

[5] The results of this section were obtained in cooperation with O. H. Nalbandyan.

is also subjected to chaotic motions of the medium, then its diffusion coefficient could be significantly changed due to the presence of a constant velocity of sedimentation. Here the tracer concentration is described by the dynamical equation (1) with constant velocity $\mathbf{v}(\mathbf{r}, t) = v - \text{const}$. As we observed earlier, this equation coincides with the one for the function $\Phi_t(\mathbf{r}) = \delta(\mathbf{r}(t) - \mathbf{r})$—the probability density of the Lagrangian particle coordinates.

Equation (1) averaged over the ensemble of **F**-realizations gives, upon application of the Furutsu–Novikov formula, Eq. (13) Taking the diffusion approximation for the variational derivative of (14), we obtain

$$\frac{\delta}{\delta F_j(\mathbf{r}', t')} q(\mathbf{r}, t) = -\exp\left\{(t-t')\left[\kappa \frac{\partial^2}{\partial \mathbf{r}^2} - \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right]\right\}\left[\delta(\mathbf{r}-\mathbf{r}')\frac{\partial}{\partial r_j} q(\mathbf{r}, t')\right]$$

In the same approximation we obtain that the function $q(\mathbf{r}, t')$ at different times t, t' is related by the evolutionary propagator

$$q(\mathbf{r}, t') = \exp\left\{-(t-t')\left[\kappa \frac{\partial^2}{\partial \mathbf{r}^2} - \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right]\right\} q(\mathbf{r}, t)$$

Consequently,

$$\frac{\delta}{\delta F_j(\mathbf{r}', t')} q(\mathbf{r}, t) = \exp\left\{\tau\left[\kappa \frac{\partial^2}{\partial \mathbf{r}^2} - \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right]\right\} \times\left[\delta(\mathbf{r}-r')\frac{\partial}{\partial \mathbf{r}_j}\exp\left\{-\tau\left[\kappa \frac{\partial^2}{\partial \mathbf{r}^2} - \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right]\right\} q(\mathbf{r}, t)\right] \quad (67)$$

where $\tau = t - t'$. Substituting (67) into (14) and performing the shift operation [operator $\exp(\tau \mathbf{v}\, \partial/\partial \mathbf{r})$], we obtain a closed-form operator equation for the mean tracer concentration (or for the probability density of the Lagrangian particle coordinate)

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\langle q(\mathbf{r}, t)\rangle = \kappa \frac{\partial^2}{\partial \mathbf{r}^2}\langle q(\mathbf{r}, t)\rangle + \frac{\partial}{\partial r_i}\int d\mathbf{r}' \int_0^t d\tau\, B_{ij}(\mathbf{r}-\mathbf{r}', \tau) \times \exp\left\{\tau\kappa \frac{\partial^2}{\partial \mathbf{r}^2}\right\}\left[\delta(\mathbf{r} - \mathbf{v}\tau - \mathbf{r}')\frac{\partial}{\partial r_j}\exp\left\{-\tau\kappa\frac{\partial^2}{\partial \mathbf{r}^2}\right\}\langle q(\mathbf{r}, t)\rangle\right] \quad (68)$$

Equation (68) can be solved explicitly by the Fourier transform. As a result, we get the mean q,

$$\langle q(\mathbf{r}, t)\rangle = \frac{1}{(2\pi)^N} \int d\mathbf{r}'\, q_0(\mathbf{r}') \int d\mathbf{q}$$

$$\times \exp\left\{ i\mathbf{q}(\mathbf{r} - \mathbf{r}' - \mathbf{v}t) - \kappa \mathbf{q}^2 t - q_i q_j \int_0^t d\tau (t-\tau)\, D_{ij}(\tau, \mathbf{q}, \mathbf{v}) \right\} \quad (69)$$

where N denotes the spatial dimension,

$$D_{ij}(\tau, \mathbf{q}, \mathbf{v}) = \int d\mathbf{k}\, E_{ij}(\mathbf{k}, \tau) \exp\{ -\kappa\tau(\mathbf{k}^2 - 2\mathbf{kq}) - i\tau\mathbf{kv}\} \quad (70)$$

and $E_{ij}(\mathbf{k}, \tau)$ is the spatial spectral density of the velocity field given by

$$B_{ij}(\mathbf{r}, t) = \int d\mathbf{k}\, E_{ij}(\mathbf{k}, t)\, e^{i\mathbf{kr}} \quad (70')$$

Furthermore, the incompressibility of the flow implies

$$E_{ij}(\mathbf{k}, t) = E(k, t)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$$

It follows from (70) that for sufficiently large $t (t \to \infty)$ one has asymptotics

$$\langle q(\mathbf{r}, t)\rangle = \frac{1}{(2\pi)^N} \int d\mathbf{r}'\, q_0(\mathbf{r}') \int d\mathbf{q}$$

$$\times \exp\{ i\mathbf{q}(\mathbf{r} - \mathbf{r}' - \mathbf{v}t) - \kappa q^2 t - q_i q_j t D_{ij}(\mathbf{v})\} \quad (71)$$

where

$$D_{ij}(\mathbf{v}) = \int_0^\infty d\tau\, D_{ij}(\tau, 0, \mathbf{v}) = \int_0^\infty d\tau \int d\mathbf{k}\, E_{ij}(\mathbf{k}, \tau)\, e^{-\kappa\tau k^2 + i\tau\mathbf{kv}} \quad (72)$$

It is clear from the expression (79) that the tensor $D_{ij}(\mathbf{v})$ is anisotropic in $\mathbf{v}$.

Let us demonstrate the validity of the *Galilean invariance principle* in our solution. The Galilean principle demands that the physical effects should be independent of a constantly moving reference frame connected to the particle. The diffusion tensor $D_{ij}(\mathbf{v})$ in (72) determined by the spectral energy density $E_{ij}(\mathbf{k}, \tau)$ was calculated in the coordinate system in which the particle has an additional drift velocity $\mathbf{v}$. If the random velocity field $\mathbf{F}(\mathbf{r}, t)$ also moves with constant velocity $\mathbf{v}$, then $\mathbf{F}(\mathbf{r}, t) = \tilde{\mathbf{F}}(\mathbf{r} - \mathbf{v}t, t)$, $\tilde{\mathbf{F}}(\mathbf{r}, t)$

being the velocity field in the moving frame. In this case the velocity correlation tensor of the velocity field is

$$B_{ij}(\mathbf{r}-\mathbf{r}', t-t') = \langle F_i(\mathbf{r}, t)\, F_j(\mathbf{r}', t')\rangle$$
$$= \langle \tilde{F}_i(\mathbf{r}-\mathbf{v}t, t)\, \tilde{F}_j(\mathbf{r}'-\mathbf{v}t')\rangle$$
$$= \tilde{F}_{ij}(\mathbf{r}-\mathbf{r}'-\mathbf{v}(t-t'); t-t')$$

Hence, by (70′),

$$E_{ij}(\mathbf{k}, \tau) = \frac{1}{(2\pi)^N}\int d\mathbf{r}\, B_{ij}(\mathbf{r}, \tau)\, e^{-i\mathbf{k}\mathbf{r}}$$
$$= \frac{1}{(2\pi)^N}\int d\mathbf{r}\, \tilde{B}_{ij}(\mathbf{r}, \tau)\, e^{-i\mathbf{k}(\mathbf{r}+\mathbf{v}\tau)}$$
$$= \tilde{E}_{ij}(\mathbf{k}, \tau)\, e^{-i\mathbf{k}\mathbf{v}\tau}$$

Here $\tilde{E}_{ij}(\mathbf{k}, \tau)$ is the velocity spectral density in the moving frame. Substituting this expression into formula (72); we see that the diffusion tensor is indeed independent of $\mathbf{v}$ in accordance with the Galilean invariance.

In this context, observe that Eq. (68) and formula (72) obtained in paper ref. 40 based on the *Bourre approximation* and the *mean-field method* (see, for example ref. 41). However, the paper arrived at the wrong conclusion, to the effect that the presence of a constant wind shift leads to the appearance of a distinguished direction and asymmetry of the diffusion. In reality, as we have just shown, the Galilean principle forbids any diffusion asymmetry.

However, one can consider a different realistic scenario whereby the particle is subject to an additional drift whereas the random field $\mathbf{F}(\mathbf{r}, t)$ is not. Such a situation arises when the diffusing particles are subjected to gravity and buoyancy forces. Notice that the mean coordinate moments

$$\langle \mathbf{r}(t)\rangle = \frac{1}{Q}\int d\mathbf{r}\, \mathbf{r}\langle q(\mathbf{r}, t)\rangle$$

determine the time evolution of the "center of mass" of the particle cloud as well as its higher moments, such as the inertia tensor of the cloud

$$\langle r_i(t)\, r_j(t)\rangle = \frac{1}{Q}\int d\mathbf{r}\, r_i r_j \langle q(\mathbf{r}, t)\rangle$$

Moreover, the total mean mass $Q = \int d\mathbf{r}\, \langle q(\mathbf{r}, t)\rangle$ is conserved. These quantities coincide with the corresponding moment functions of the particle position. In particular, we obtain the following relations for $\langle \mathbf{r}(t) \rangle$ and $\sigma_{ij}^2 = \langle [r_i(t) = \langle r_i(t) \rangle][r_j(t) = \langle r_j(t) \rangle] \rangle$:

$$\langle \mathbf{r}(t) \rangle = \mathbf{r}(0) + \mathbf{v}t, \qquad \frac{d}{dt} \sigma_{ij}^2(t) = 2 \left[\kappa \delta_{ij} + \int_0^t d\tau \, D_{ij}(\tau, 0, \mathbf{v}) \right]$$

Consequently, for large times t, the quantity $D_{ij}(\mathbf{v})$ determines the *turbulent diffusion coefficient*,

$$D_{ij}^{\mathrm{tur}} = \lim_{t \to \infty} \frac{d}{dt} \sigma_{ij}^2(t) = 2[\kappa \delta_{ij} + D_{ij}(\mathbf{v})]$$

Furthermore,

$$D_{ij}(\mathbf{v}) = A(\mathbf{v}) \, \delta_{ij} + B(\mathbf{v}) \, \Delta_{ij}(\mathbf{v})$$

with coefficients

$$A(\mathbf{v}) = \frac{v_i v_j}{\mathbf{v}^2} D_{ij}(\mathbf{v}), \qquad B(\mathbf{v}) = \frac{1}{N-1} \left[\delta_{ij} - N \frac{v_i v_j}{\mathbf{v}^2} \right] D_{ij}(\mathbf{v})$$

The above representation has the following implication: if one of the coordinate axes is aligned along the $\mathbf{v}$-vector, then the particle's diffusion is statistically independent of the transverse directions, and the diffusion coefficient in the $\mathbf{v}$ direction is determined by $D_{\|} = A(\mathbf{v})$. Furthermore, the transverse diffusion coefficient is $D_{\perp} = A(\mathbf{v}) + B(\mathbf{v})$. This property is directly related to the finitude of the time correlation radius of the random velocity field $\mathbf{F}(\mathbf{r}, t)$ and is absent in the delta-correlated case. In the new coordinate system, the formula (71) assumes the form

$$\begin{aligned} \langle q(\mathbf{r}, t) \rangle = \frac{1}{(2\pi)^N} \int d\mathbf{r}' \, q_0(\mathbf{r}') \int dq \int d\mathbf{q}_{\perp} \\ \times \exp\{ iq(x - x' - vt) + q\mathbf{q}_{\perp}(\boldsymbol{\rho} - \boldsymbol{\rho}') \\ - q^2 t[\kappa + D_{\|}(\mathbf{v})] - q_{\perp}^2 t[\kappa + D_{\perp}(\mathbf{v})] \} \end{aligned}$$

or

$$\begin{aligned} \langle q(\mathbf{r}, t) \rangle = \frac{1}{[4\pi t(\kappa + A + B)]^{N/2}} \left(\frac{\kappa + A + B}{\kappa + A} \right)^{1/2} \int d\mathbf{r}' \, q_0(\mathbf{r}') \\ \times \exp \left\{ \frac{-(x - x' - vt)^2}{4t[\kappa + D_{\|}(\mathbf{v})]} - \frac{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2}{4t[\kappa + D_{\perp}(\mathbf{v})]} \right\} \end{aligned} \tag{71'}$$

To evaluate the diffusion coefficients we will use the model of spectral density

$$E(k, \tau) = E(k) \exp(-|\tau|/\tau_0)$$

where τ_0 is the time correlation radius of the random velocities. In this case,

$$D_{ij}(\mathbf{v}) = \frac{1}{v} \int d\mathbf{k}\, E(k)\, \Delta_{ij}(\mathbf{k}) \frac{p}{k} \frac{1}{1 + p^2(\mathbf{kv})^2/k^2v^2}$$

where the parameter is

$$p(\mathbf{k}, \mathbf{v}) = \frac{kv\tau_0}{1 + \kappa k^2 \tau_0}$$

Consequently, in the 3D case we arrive at the formulas for turbulent diffusion rates,

$$D_{\|}(v) = \frac{4\pi}{v} \int_0^\infty dk\, kE(k)\, f_{\|}(k, v)$$

$$D_{\perp}(v) = \frac{4\pi}{v} \int_0^\infty dk\, kE(k)\, f_{\perp}(k, v)$$

with functions

$$f_{\|}(k, v) = \left[\arctan p + \frac{1}{p} \left(\frac{1}{p} \arctan p - 1 \right) \right]$$

$$f_{\perp}(k, v) = \frac{1}{2} \left[\arctan p - \frac{1}{p} \left(\frac{1}{p} \arctan p - 1 \right) \right]$$

If p is small (i.e., when $v\tau_0 \ll l_0$, where l_0 is the spatial correlation radius of the velocity field), the functions $f_{\|}(p)$ and $f_{\perp}(p)$ are close to $2p/3$, which corresponds to isotropic diffusion, independent of the sedimentation velocity v. For large values of p (i.e., when $v\tau_0 \gg l_0$) we have $f_{\|}(p) = 2f_{\perp}(p) \approx \pi/2$. Such an anisotropic diffusion is explained by the fact that the tracer mixing by the turbulent flow decreases the time the tracer particles spend within the velocity correlation radius. Additionally, in the isotropic field of random velocities, the transverse correlation radius of the velocity field is only half of the longitudinal correlation radius,[1] which explains the above anisotropy of the diffusion coefficient. For parameter values $\kappa\tau_0 \ll l_0^2$, the diffusion tensor $D_{ij}(\mathbf{v})$ does not depend on parameter κ.

We have demonstrated that anisotropic diffusion of the tracer undergoing sedimentation is essentially connected with the finitude of the correlation radius. Furthermore, using the diffusion approximation and Eulerian description of tracer clouds with characteristic length scale r_0, we found the following limits of its applicability:

$$D_{\|}(v)\,\tau_0 \ll r_0^2, \qquad D_{\perp}(v)\,\tau_0 \ll r_0^2$$

The latter, however, are valid only for a sufficiently small time correlation radius. For small fluctuations of the velocity field this restriction is not essential.

6. CONCLUSIONS

We utilize a unified functional approach to study the statistics of passive tracer diffusion in Gaussian random incompressible velocity fields, and provide both the general setup of the problem and two approximate methods of analysis.

The general problem is very complicated. It contains many parameters: the mean flow, statistics of fluctuations, molecular diffusion rate, etc. Their combined effect could not be adequately expressed by the mean tracer concentration or its correlation function. Rather, one needs to study the statistics of the problem at the level of joint probability densities of the tracer concentration and of its spatial gradient. Even in the simplest case of zero mean flow with zero molecular diffusion, in the delta-correlated approximation, the time evolution of the tracer fluctuations is still rather complicated. The initially smooth tracer distribution becomes more and more spatially chaotic and disordered, its spatial gradients show strong blowup, and the level lines of concentration assume fractal-like features. In addition, this process moves ill the direction of decreasing spatial scales, so eventually one arrives at scales of molecular diffusion. At this stage one loses the closed-form statistical description of probability densities. So one has to work out various approximate schemes. The first efforts of this kind are found in refs. 42–45. The quantitative description of effects of mean flow (even in the simplest case of parallel shear flow) and specific models of velocity field fluctuations (for example, the correlation tensor of the fluctuating compŏnent) can be analyzed on the basis of numerical solutions of the corresponding Fokker–Planck equations and computer simulations.

The functional approach presented in this paper is based essentially on the assumption of a finite time correlation radius of the velocity field. The conditions of applicability of different approximation schemes are expressed in terms of the correlation radius. Our results do not apply to fields with large or infinite correlation radius, such as a *stationary* (*in time*) *random velocity field*. The latter case has not been studied yet in any generality, although some partial problems of this type have been considered.[46,47]

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